Ergodic Theory - Week 10

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1 Spectral theory of measure-preserving systems

P1. Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and let $f \in L^2(X)$. Let μ_f be the spectral measure of f and let $P_T f$ be the orthogonal projection onto the closed subspace of T-invariant functions in $L^2(X)$. Show that:

(a) Show that for every $t \in \mathbb{T}$, we have

$$\mu_f(\{t\}) = \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} e(-nt) \int \bar{f} \cdot T^n f \, d\mu.$$

Conclude that $\mu_f(\{0\}) = \|P_T f\|_2^2$.

(b) If the system is ergodic, then show that $\mu_f(\{0\}) = 0$ if and only if $\int f d\mu = 0$.

(c) Show that $\mu_f(\mathbb{T}) = ||f||_2^2$.

P2. Let $\{a(n)\}_{n \in \mathbb{N}}$ be a sequence in $\mathbb{N}$ such that for every $\beta \in (0, 1)$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e(a(n)\beta) = 0.$$

Suppose (X, μ, T) is a measure-preserving system. Let $f \in L^2(\mu)$. Prove that

$$\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} T^{a(n)} f - P_T f \right\|_{L^2(\mu)} = 0,$$

where P_T denotes the projection to the space of invariant functions.

P3. Let v be any continuous (that is $v(\{x\}) = 0$ for all $x \in \mathbb{T}$) Borel probability measure on the torus $\mathbb{T}$ that is invariant under the maps $T_p x = px \pmod 1$ for all $p \in \mathbb{N}$. Show that v is the Lebesgue measure (a famous conjecture of Furstenberg asserts that the same conclusion holds under the much weaker assumption that the measure is invariant under T_2 and T_3).

Hint: For a finite Borel measure μ on $\mathbb{T}$, we define its Fourier coefficients by $\widehat{\mu}(n) = \int e(nt) \, d\mu(t)$. Use the fact that if two measures have the same Fourier coefficients, then they are equal.

P4. **Optional:** In this exercise, we outline the steps to prove Sárközy's theorem: if $(X, \mathcal{B}, \mu, T)$ is a measure-preserving system and $A \in \mathcal{B}$ has positive measure, then there exist infinitely many $n \in \mathbb{N}$ such that $\mu(A \cap T^{-n^2} A) > 0$.

This can be combined with a well-known argument of Furstenberg to prove that if a set $E \subseteq \mathbb{N}$ has positive density, then we can always find $x, y \in E$ such that $x - y$ is a perfect square. In fact, the same proof works for any integer polynomial with zero constant term, like $n^3, n^4 + 4n^3$, etc.

(a) Let $f = \mathbb{1}_A$ and let μ_f be the spectral measure of f. Show that

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int \overline{f} \cdot T^{n^2} f \, d\mu = \mu_f(\{0\}) + \sum_{q\in\mathbb{Q}\cap[0,1]} \mu_f(\{q\}) S_q,$$

where

$$S_q = \lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} e(qn^2).$$

Hint: Rewrite the left-hand side in terms of the spectral measure of f and use Weyl's theorem to eliminate the contribution of the irrationals in the integral.

(b) Repeat the previous step to show that if W is any positive integer, then

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int \overline{f} \cdot T^{(Wn)^2} f \, d\mu = \sum_{q\in A_1} \mu_f(\{q\}) + \sum_{q\in(\mathbb{Q}\cap[0,1])\setminus A_1} \mu_f(\{q\}) S_{W,q} \tag{1}$$

where A_1 consists of the rationals $q \in \mathbb{Q}$ for which $W^2 q$ is an integer and

$$S_{W,q} = \lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} e(q(Wn)^2).$$

(c) Show that we can pick W sufficiently large to make the contribution of the second sum negligible. Namely, for every $\varepsilon > 0$, show that we can pick $W \in \mathbb{N}$, such that

$$\lim_{N\to+\infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} \int \overline{f} \cdot T^{(Wn)^2} f \, d\mu \right| \geq \mu_f(\{0\}) - \varepsilon.$$

Conclude that $\mu(A \cap T^{-(Wn)^2} A) > 0$ for infinitely many $n \in \mathbb{N}$.